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Critically indecomposable graphs

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ABSTRACT

An undirected graph $G = (V, E)$ with a specific subset $X \subset V$ is called X-critical if G and $G(X)$, induced subgraph on X, are indecomposable but $G(V - \{w\})$ is decomposable for every $w \in V - X$. This is a generalization of critically indecomposable graphs studied by Schmerl and Trotter [J.H. Schmerl, W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, Discrete Mathematics 113 (1993) 191–205] and Bonizzoni [P. Bonizzoni, Primitive 2-structures with the $(n-2)$-property, Theoretical Computer Science 132 (1994) 151–178], who deal with the case where X is empty.

We present several structural results for this class of graphs and show that in every X-critical graph the vertices of $V - X$ can be partitioned into pairs (a_1, b_1), $(a_2, b_2), \ldots, (a_m, b_m)$ such that $G(V - \{a_{j_1}, b_{j_1}, \ldots, a_{j_k}, b_{j_k}\})$ is also an X-critical graph for arbitrary set of indices $\{j_1, \ldots, j_k\}$. These vertex pairs are called *commutative elimination sequence*. If G is an arbitrary indecomposable graph with an indecomposable induced subgraph $G(X)$, then the above result establishes the existence of an indecomposability preserving sequence of vertex pairs $(x_1, y_1), \ldots, (x_t, y_t)$ such that $x_i, y_i \in V - X$. As an application of the commutative elimination sequence of an X-critical graph we present algorithms to extend a 3-coloring (similarly, 1-factor) of $G(X)$ to entire G.

1. Introduction

An undirected graph G is a 2-tuple (V, E), where V is called the vertex set and E, which is a collection of unordered pairs of vertices, is called the edge set. For a subset $X \subseteq V$, $G(X) = H(X, E')$ denotes the induced subgraph of G on subset X with $E' = \{(u, v) \in E : u, v \in X\}$. Henceforth $G = (X, E)$ denotes an undirected graph.

A *module (or interval)* (Fräissé [3]) of an undirected graph $G = (V, E)$ is a subset of vertices, $M \subseteq V$ such that for any $a, b \in M$ and $c \in V - M$, $(a, c) \in E$ if $(b, c) \in E$. This definition is interesting if $1 < |M| < |V|$, otherwise M is called a *trivial* module. A graph is called *indecomposable* (or *prime*, *base-level*) if it has only trivial modules. A non-trivial module is *maximal*, if it is not contained in any other non-trivial module. Given any graph, one can replace the maximal modules by single vertices to get an indecomposable graph. Numerous graph problems can be solved for general graphs if one can find the solution on indecomposable graphs. These include the problems in domination, matching, coloring, optimal spanning tree, and graph isomorphism. This is why the study of indecomposable graphs has attracted significant interest from mathematicians and computer scientists (see [4–8]).

Ehrenfeucht and Rozenberg [9,10] have studied **2-structures** (2S), which are a generalization of graphs in the following sense. In a graph ordered/unordered pairs of vertices are partitioned into two sets: edges and non-edges but in 2-structures these are partitioned into arbitrary number of classes. The concept of a module generalizes naturally to 2S, which they refer to as *clan*. The 2S without a non-trivial clan, i.e., indecomposable 2S is called a *primitive* 2S. Based on their work it appears that

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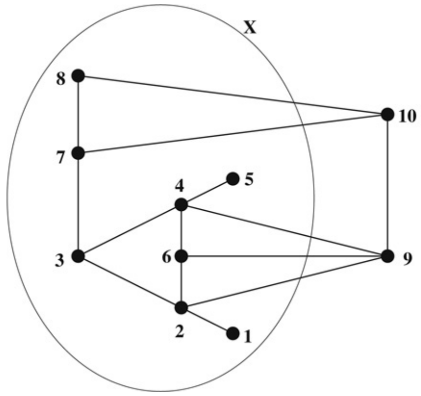

Fig. 1. An X-critical graph.

most of the properties of indecomposable graphs generalize to 2S. A monograph by Ehrenfeucht, Harju and Rozenberg [11] presents a vast cache of very interesting properties of 2-structures, especially primitive 2-structures.

Significant literature exists on the structural properties of indecomposable graphs and how indecomposability is inherited by induced subgraphs. In this regard, the concept of criticality has been studied. In an indecomposable graph a vertex is said to be *critical* if the graph turns decomposable on the removal of that vertex.

A seminal work in this direction is by Schmerl and Trotter [1] and independently by Bonizzoni [2]. Both works are for general 2-structures, while Schmerl and Trotter do not refer to it by that name. For simplicity, this work considers graphs instead of more general 2-structures, therefore we will (re)state existing results only in the context of graphs. The aforementioned papers deal with *critically indecomposable graphs*, in which every vertex is critical. Bonizzoni refers to critical indecomposability as $(n-2)$-*property* because the largest indecomposable subgraphs of such a graph have $|V|-2$ vertices. They showed that for all $m \geq 2$, the bipartite graphs on vertex set $\{u_1, \dots, u_m\} \cup \{v_1, \dots, v_m\}$, where each u_j is adjacent to each v_k if $j \leq k$, is critically indecomposable. The only other graphs which are also critically indecomposable are the complement of these graphs. In this work we consider a natural generalization of the notion of critical indecomposability.

An indecomposable graph $G = (V, E)$ is X-*critical* for $X \subset V$, if $G(X)$ is indecomposable and each vertex of $V - X$ is critical. See Fig. 1 for an example. As is obvious from the definition, every indecomposable graph is X-critical for some X. The critically indecomposable graphs defined by Schmerl and Trotter are $\emptyset$-critical. **Throughout this paper we will assume that** $|X| \geq 4$, because indecomposability on smaller set of vertices cannot be defined meaningfully. An important theorem by Schmerl and Trotter, and by Ehrenfeucht and Rozenberg follows.

Theorem 1 (*[10,9,1]*).

(a) *Let $G = (V, E)$ be an indecomposable graph with an indecomposable subgraph $G(X)$ such that $4 \leq |X| \leq |V| - 2$. Then there exists a pair of distinct vertices $a, b \in V - X$ such that $G(X \cup \{a, b\})$ is indecomposable.*
(b) *If $G = (V, E)$ is an indecomposable graph such that $|V| \geq 7$, then there is $V' \subset V$ such that $|V - V'| = 2$ and $G(V')$ is indecomposable.*

The motivation for studying the structure of X-critical graphs comes from a very interesting result proved by Ille [12], which generalizes part (b) of Theorem 1.

Theorem 2 (*[12]*). *Let $G = (V, E)$ be an indecomposable graph and there be a set $X \subseteq V$ of vertices satisfying $|V - X| \geq 6$ such that $G(X)$ is also indecomposable, then there exist two vertices $a, b \in V - X$ such that $G(V - \{a, b\})$ is also indecomposable.*

If $|V - X|$ is even, then Theorem 2 can be deduced from Theorem 1(a). Otherwise, from the same result, we can find a vertex $x \in V - X$ such that $G(V - \{x\})$ is indecomposable. Additionally, if there exists a $y \in V - (X \cup \{x\})$ such that $G(V - \{x, y\})$ is indecomposable, then again we satisfy the claim in Theorem 2. The trouble seems to be in the case when no such y exists, in other words, $G(V - \{x\})$ is X-critical.

In this work we study the X-critical graphs and prove some structural theorems and give an efficient algorithm to compute a sequence of vertex pairs which can be removed without disturbing the X-criticality property. These results give an alternative proof of Theorem 2. We also give efficient polynomial algorithms for computing perfect matching and 3-coloring for X-critical graphs.

2. Critical indecomposability and related notions

This work only considers undirected graphs, but it can be generalized to directed graphs. Symmetric 0-1 matrix e will represent adjacency. So, $e_{uv} = 1$ if and only if edge (u, v) belongs to E.

Definition 3. Let $G = (V, E)$ and $Y \subset V$ and $x \in Y$. If $G(Y)$ is indecomposable but $G(Y - \{x\})$ is decomposable, then x is said to be *critical* in Y.

A property of modules trivially deducible from the definition is as follows.

Observation 1. (i) *M is a module of G and $Y \subseteq V$. Then $M \cap Y$ is a module of $G(Y)$.*
(ii) *If M_1 and M_2 are modules of G such that $M_1 \cap M_2$ is non-empty then $M_1 \cup M_2$ is also a module of G.*

Definition 4. Let $G = (V, E)$, $Y \subset V$, and $x \in V - Y$. If $\forall y \in Y$ $(x, y) \in E$ or $\forall y \in Y$ $(x, y) \notin E$, then x is said to be *global* to Y.

The definition of a module can be stated in terms of this concept as follows. The vertex set M in G is a module if and only if each $x \in V - M$ is global to M. A trivial observation follows.

Observation 2. *Let M be a module in $G = (V, E)$ and V_1 be a subset of V. If a vertex $x \in M$ is global to $V_1 - M$. Then $V_1 - M$ is a module in $G(V_1)$.*

Definition 5. A graph $G = (V, E)$ is said to be *marginally decomposable* if (i) there is only one non-trivial module in the graph and, (ii) the size (vertex cardinality) of the module is either 2 or $|V| - 1$.

Corollary 6. *Let G be a marginally decomposable graph with M being its unique non-trivial module. If there is a vertex $x \in M$ which is global to $V - M$, then $|V - M| = 1$.*

Next we define a notion which is more stringent than X-criticality.

Definition 7. Let $G = (V, E)$ be a graph and $X \subseteq V$. Then G is said to be *X-stably-indecomposable* (in short X-stable) if (i) G and induced subgraph $G(X)$ are indecomposable, and (ii) $G(V - \{w\})$ is marginally decomposable for all $w \in V - X$.

It is obvious that every X-stable graph is X-critical. In Section 3 we will establish that the two concepts are equivalent.

Lemma 8. *In graph $G = (V, E)$, let Y be a subset of V with 5 or more vertices and $x \in Y$. Then $G(Y - \{x\})$ is indecomposable and $G(Y)$ is decomposable iff $G(Y)$ is marginally decomposable with the non-trivial module $Y - \{x\}$ or $\{x, y\}$ for some $y \in Y - \{x\}$.*

Proof. (Only if) Let M be a non-trivial module of $G(Y)$. Subgraph $G(Y - \{x\})$ is indecomposable so $M' = M \cap (Y - \{x\})$ must be a trivial module of $G(Y - \{x\})$. Thus M' can be either $Y - \{x\}$ or y for some $y \in Y - \{x\}$. Thus M will be $\{x, y\}$ or $Y - \{x\}$. Next we show that at most one such M is possible.

Let M_1 and M_2 be non-trivial modules of $G(Y)$. There are two cases to be considered: (i) $M_1 = \{x, y_1\}$, $M_2 = \{x, y_2\}$ and (ii) $M_1 = \{x, y\}$, $M_2 = Y - \{x\}$. In case (i), $\{y_1, y_2\}$ is a module of $G(Y - \{x\})$ and in case (ii) $Y - \{x, y\}$ is a module of $G(Y - \{x\})$. In each case the module is non-trivial so it contradicts the fact that $G(Y - \{x\})$ is indecomposable.

(If) Consider the case where $\{x, y\}$ is the module of $G(Y)$. Assume that M is a non-trivial module of $G(Y - \{x\})$. If $y \in M$ then $M \cup \{x\}$ is a non-trivial module of $G(Y)$. Uniqueness requires that $M \cup \{x\} = \{y, x\}$ thus $M = \{y\}$, i.e., M is trivial. If $y \notin M$, then M is also a non-trivial module of $G(Y)$. In this case uniqueness requires that $M = \{x, y\}$ which is also not possible since $y \notin M$.

Next consider the case of module $Y - \{x\}$. In this case x is global to $Y - \{x\}$. Then it is global to any subset M of $Y - \{x\}$. If M is a module of $G(Y - \{x\})$, then it must also be a module of $G(Y)$. Thus $M = Y - \{x\}$, but this is a trivial module of $G(Y - \{x\})$. □

Lemma 9. *Let $G = (V, E)$ be X-critical and subgraph $G' = G(V - \{a, b\})$ be indecomposable for some $a, b \in V - X$. Then $G(V - \{a, b\})$ is also X-critical.*

Proof. Suppose G' is not X-critical. So there exists $c \in V - X - \{a, b\}$ such that $G(V - \{a, b, c\})$ is also indecomposable. Since $|V - \{a, b, c\}| \geq |X| \geq 4$, we can use Theorem 1(a) to deduce that there are u, v in $\{a, b, c\}$ such that $G(V - \{a, b, c\} \cup \{u, v\})$ is indecomposable. This graph is $G'' = G(V - \{w\})$ where w is one of a, b, c. On the contrary, by the definition of X-critical graph, G'' is decomposable. □

Definition 10. If G is X-critical and $a, b \in V - X$ such that $G(V - \{a, b\})$ is indecomposable (thus X-critical) then the unordered pair (a, b) will be called a locked pair of G.

Lemma 11. *Let $G = (V, E)$ be X-critical and $V - X$ be non-empty. Then G has a locked pair.*

Proof. Consider the indecomposable subgraph $G(X)$. From Theorem 1(a) we know that there is an indecomposable subgraph $G(Y)$ such that $X \subset Y$ and $|Y| = |X| + 2$. Repeating the argument we find that there is an indecomposable subgraph $G(V')$ such that $X \subseteq V'$ and $1 \leq |V - V'| \leq 2$. But $|V - V'|$ cannot be 1 since G is X-critical and V' contains X. Suppose $V' = V - \{a, b\}$. From Lemma 9 we conclude that (a, b) is a locked pair in G. □

An X-critical subgraph cannot have vertex cardinality equal to $|X| + 1$ because of the criticality condition. Combining this fact with Lemma 11 leads to the following corollary.

Corollary 12. *$G = (V, E)$ is an X-critical graph, then $|V - X| = 2k$ for some $k \geq 0$.*

Let $G = (V, E)$ be a graph and $X \subset V$ such that $G(X)$ is indecomposable. Then, from Lemma 8, for any vertex $y \in V - X$, only one of the following three cases are possible: (i) $G(X \cup \{y\})$ is indecomposable, (ii) $G(X \cup \{y\})$ is decomposable with the unique non-trivial module $\{y, z\}$ for some $z \in X$, and (iii) $G(X \cup \{y\})$ is decomposable with the unique non-trivial module X, i.e., y is global to X. We partition the vertices of $V-X$ based on these cases. If it is case (i), then y belongs to a class denoted by $extn(X)$, in case of (ii) y belongs to a class denoted by $eq_X(z)$, finally in the third case y belongs to a class denoted by $[X]$. We denote this partition by $\mathcal{C}(V-X, X)$. This terminology is adopted from [7,13]. By eq_X we shall denote the union of all $eq_X(z)$ classes.

Let G be an X-critical graph and (a, b) be a locked pair. Both $G(V-\{a\})$ and $G(V-\{b\})$ are decomposable so neither vertex can belong to $extn(V-\{a, b\})$. Further, both vertices cannot belong to $[V-\{a, b\}]$ because that would imply that $V-\{a, b\}$ is a module of G which is absurd since G is indecomposable.

Observation 3. *Let (a, b) be a locked-pair in an X-critical graph $G=(V, E)$. Then either* (i) *both a and b are in class $eq_{V-\{a,b\}}$ or* (ii) *one each is in $eq_{V-\{a,b\}}$ and $[V-\{a, b\}]$.*

3. Structural theorem

The main goal of this section is to prove that X-critical and X-stable are equivalent properties. Every X-stable is trivially X-critical so we only need to prove that X-critical implies X-stable. We shall establish this claim by induction on the number vertices in $V-X$. Consider an X-critical G. If $|V-X|=0$, then $G=G(X)$. In this case G is trivially X-critical and X-stable. Next, let $|V-X|>0$. From Lemma 11, G has a locked pair (a, b). So $G(V-\{a, b\})$ is X-critical. From induction hypothesis this graph is X-stable. To complete the proof we need to show that G is also X-stable.

Based on Observation 3 the proof is split into two cases.

3.1. Case of $a \in [V-\{a, b\}]$ and $b \in eq_{V-\{a,b\}}$

In this subsection we consider the case where $\{b, p\}$ is the module of $V-\{a\}$ and $V-\{a, b\}$ is the module of $V-\{b\}$. Thus $e_{ab} \neq e_{az}$ for any $z \in V-\{a, b\}$ because a cannot be global to $V-\{a\}$. We have a trivial observation.

Observation 4. *Let M be a module of $V-\{w\}$ for any $w \in V-\{a, b\}$. Then* (i) *If $b \notin M$ then $M-\{a\}$ is a module of $V-\{w\}$ and* (ii) *If $b \in M$ then $a \in M$.*

Lemma 13. *If $p \notin X$, then $G(V-\{p\})$ is marginally decomposable and its module is $\{a, u\}$ where u is some vertex in $V-\{a, b, p\}-X$.*

Proof. Let M be a non-trivial module of $G(V-\{p\})$. If a and b both belong to M, then $M \cup \{p\}$ will become a non-trivial module of G, because $\{b, p\}$ is a module of $G(V-\{a\})$, i.e., b and p have same connectivity with $V-\{a, b, p\}$. If neither of the two vertices belongs to M, then it is also a non-trivial module of G because $e_{mb}=e_{mp} \forall m \in M$ due to $a \notin M$. Neither of these cases is possible because G is indecomposable. The case of $b \in M$ and $a \notin M$ is eliminated by Observation 4(ii). So we must have $a \in M$ and $b \notin M$.

From Observation 4(i), $M-\{a\}$ is also a module of $G(V-\{p\})$. As $\{b, p\}$ is a module of $G(V-\{a\})$, $M-\{a\}$ is also a module of G. The indecomposability of G implies that $|M-\{a\}|=1$. So $M=\{a, u\}$, where $u \in V-\{a, b, p\}$.

Vertex a is global to $V-\{a, b\}$ and $\{a, u\}$ is a module of $G(V-\{p\})$ so u is global to $V-\{a, b, p, u\}$. In particular it is global to $X-\{u\}$. If $u \in X$, then $X-\{u\}$ is a module of $G(X)$, which is absurd because $G(X)$ is indecomposable. So $u \notin X$.

To complete the proof we will show that $G(V-\{p\})$ cannot have more than one non-trivial module. Let $\{a, u_1\}$ and $\{a, u_2\}$ be two of its modules. Then $\{a, u_1, u_2\}$ is also a module of $G(V-\{p\})$. But this contradicts our conclusion in the last paragraph that all its modules are of the form $\{a, u\}$. □

Corollary 14. *If $p \notin X$, then there exists $u \in V-\{a, b, p\}-X$ which is global to $V-\{b, p, u\}$.*

Recall that $G(V-\{a, b\})$ is marginally indecomposable so $G(V-\{a, b, w\})$ has a unique non-trivial module for all $w \in V-\{a, b\}$.

Corollary 15. *If $p \notin X$, then the unique non-trivial module of $G(V-\{a, b, p\})$ is $V-\{a, b, p, u\}$ where u is some vertex of $V-\{a, b, p\}-X$.*

Corollary 16. *If $w \in V-\{a, b, p\}-X$, then $V-\{a, b, w, p\}$ cannot be a module of $V-\{a, b, w\}$.*

Proof. If $V-\{a, b, w, p\}$ is a module of $V-\{a, b, w\}$, then p is global to $V-\{a, b, w, p\}$. So $p \notin X$ because otherwise $X-\{p\}$ would be a module of $G(X)$, which is indecomposable. From the lemma, there exists $u \in V-\{a, b, p\}-X$ such that $\{a, u\}$ is a module of $V-\{p\}$. Vertex a is global to $V-\{a, b\}$ so u is global to $V-\{a, b, p, u\}$. If $u \neq w$, then owing to the global nature of u and p, $V-\{a, b, w\}$ has a second module, namely, $V-\{a, b, w, p, u\}$ which is non-trivial since it contains X. If $u=w$, then w is global to $V-\{a, b, w, p\}$. Hence $V-\{a, b, w, p\}$ is a module of $V-\{a, b\}$, which is also non-trivial. Both situations are impossible because $V-\{a, b, w\}$ is marginally decomposable and $V-\{a, b\}$ is indecomposable. □

Corollary 17. *Let M be a non-trivial module of $V-\{w\}$ for some $w \in V-X-\{a,b,p\}$. If $a \notin M$, then $M=\{x,z\}$, where $x \in V-\{a,b,w\}$ and $z \in V-\{a,b,w,x\}$.*

Proof. Given that a does not belong to M, from Observation 4(ii) we know that b also does not belong to M. Thus M is contained in $V-\{a,b,w\}$ and consequently it is its module. From induction hypothesis $V-\{a,b,w\}$ is marginally decomposable so the size of M can be 2 or $|V-\{a,b,w\}|-1$ or $|V-\{a,b,w\}|$. In the former case $M=\{x,z\}$ where $x \in V-\{a,b,w\}$ and $z \in V-\{a,b,w,x\}$. In the second case $M=V-\{a,b,w,z\}$. From Corollary 16, z cannot be p. As b is out of M, it is global to M. This implies that p is global to $V-\{a,b,w,z,p\}$. Therefore $V-\{a,b,w,z,p\}$ is a module of $V-\{a,b,w\}$, which is not possible due to its size.

Finally consider the case of $M=V-\{a,b,w\}$. Then b is global to M, which in turn, implies that p is global to $V-\{a,b,w,p\}$. So $V-\{a,b,w,p\}$ is a module of $V-\{a,b,w\}$ in contradiction to Corollary 16. □

Lemma 18. *Let M be a non-trivial module of $V-\{w\}$ for some $w \in V-\{a,b,p\}$. If $a \in M$, then $M=V-\{w,z\}$ for some $z \in V-\{a,b,p,w\}$.*

Proof. $M_1'=M\cap(V-\{a,b,w\})$ is a module of $V-\{a,b,w\}$. It is given that a is global to $V-\{a,b\}$ so $M_2'=V-\{a,b,w\}-M$ is also a module of $V-\{a,b,w\}$ from Observation 2. The two are distinct so at least one of them is trivial because $V-\{a,b,w\}$ is marginally decomposable. We conclude that either $|(V-\{a,b,w\})\cap M|\le 1$ or $|(V-\{a,b,w\})\cap M|\ge |V-\{a,b,w\}|-1$.

Case of $|M\cap(V-\{a,b,w\})|\le 1$:

In this case the only possible value of M is either $\{a,b\}$ or $\{a,z\}$ or $\{a,b,z\}$ where $z \in V-\{a,b,w\}$, because of Observation 4(ii) and the fact that M is non-trivial. In the first case b is global to $V-\{a,b,w\}$, so p is global to $V-\{a,b,p,w\}$. In the second and third cases if $z=p$ then again p is global to $V-\{a,b,p,w\}$. In both these cases $V-\{a,b,p,w\}$ is a module of $V-\{a,b,w\}$, which is not possible due to Corollary 16. This leaves the possible form of M to be $\{a,z\}$ or $\{a,b,z\}$ where $z \in V-\{a,b,w,p\}$. In the former case we have $e_{zb}=e_{ab}\neq e_{ap}=e_{zp}=e_{zb}$. Thus this case is also impossible. In the latter case $e_{az}=e_{ap}=e_{bp}=e_{zp}=e_{zb}$. This implies $\{a,b\}$ must also be a module of $V-\{w\}$, but that is shown above to be impossible.

Case of $|(V-\{a,b,w\})\cap M|\ge|V-\{a,b,w\}|-1$:

In this case, due to Observation 4(ii), possible values of M are $V-\{b,w\}$, $V-\{a,b,w\}$, $V-\{b,z,w\}$, and $V-\{z,w\}$ for some $z \in V-\{a,b,w\}$. In the first and the second cases; and in the third and the fourth cases with $z=p$, p is global w.r.t. $V-\{a,b,p,w\}$ which is not possible since it implies that $V-\{a,b,p,w\}$ is a module of $V-\{a,b,w\}$ contradicting Corollary 16. In the third case with $z\neq p$, both p and z are global w.r.t. $V-\{a,b,p,z,w\}$ because $\{b,p\}$ is a module of $V-\{a\}$. Thus $V-\{a,b,p,z,w\}$ is a module of $V-\{a,b,w\}$, which is not possible due to its size. Therefore only possible form of M is $V-\{z,w\}$ where $z \in V-\{a,b,p,w\}$. □

Lemma 19. *Let $G=(V,E)$ be X-critical. Let (a,b) be a locked pair with $a \in [V-\{a,b\}]$ and $b \in eq_{(V-\{a,b\})}$. If $G(V-\{a,b\})$ is X-stable, then G is also X-stable.*

Proof. G is X-critical so $G(V-\{w\})$ has at least one module for each $w \in V-X$. Hence we have to show that the module is unique and its size is either 2 or $|V|-2$. The case of $w=p$ is settled in Lemma 13. The case $w=a$ or $w=b$ is also easy because, from the definition of the locked pair and Lemma 8, $V-\{a\}$ and $V-\{b\}$ are marginally decomposable.

Finally let us consider the case of $w \in V-\{a,b,p\}$. From Corollary 17 and Lemma 18, the only possible modules of $V-\{w\}$ are: $\{x,z\}$ and $V-\{w,y\}$, where $x \in V-\{a,b,w\}$, $z \in V-\{a,b,w,x\}$, and $y \in V-\{a,b,p,w\}$. The unique module of $V-\{a,b,w\}$ in these cases is respectively $\{x,z\}$ and $V-\{a,b,w,y\}$. Since $|X|\ge 4$ and $a,b,w \notin X$, $|V-\{a,b,w,y\}|\ge 3$. Hence each such module of $V-\{a,b,w\}$ is distinct. If $V-\{w\}$ has more than one module, then $V-\{a,b,w\}$ must also have more than one module. This contradicts the fact that $V-\{a,b,w\}$ is marginally decomposable. □

3.2. Case of $a, b \in eq_{V-\{a,b\}}$

Now we will consider the case where $\{a,q\}$ is the module of $V-\{b\}$ and $\{b,p\}$ is the module of $V-\{a\}$. Vertices p and q must be distinct because otherwise $\{a,b,p\}$ will be a module of G which is an indecomposable graph. In this case we have $e_{ab}\neq e_{ap}=e_{qp}=e_{qb}$: the first inequality is because otherwise $\{b,p\}$ will become a module in G which is indecomposable, the following equality is because $\{a,q\}$ is a module of $V-\{b\}$ and the last one is based on the fact that $\{b,p\}$ is a module of $V-\{a\}$.

We need to show that $G(V-\{w\})$ is also marginally decomposable for all $w \in V-X$. G being X-critical, it is sufficient to show that it does not have more than one module.

3.2.1. Sub-case $w \in \{a,b\}$

G is X-critical so $G(V-\{a\})$ is decomposable. $G(V-\{a,b\})$ is indecomposable so from Lemma 8 we know that $G(V-\{a\})$ must be marginally decomposable. For a similar reason $G(V-\{b\})$ is also marginally decomposable.

3.2.2. Sub-case $w = p$ where $p \notin X$

The only module of $G(V - \{a\})$ is $\{b, p\}$ so $G(V - \{a, b\})$ is isomorphic to $G(V - \{a, p\})$ under the mapping $x \to x$ for all $x \in V - \{a, b, p\}$ and $p \to b$. Thus $G(V - \{a, p\})$ is also an X-critical graph because the image of X under this mapping is X itself. From Lemma 8, $G(V - \{p\})$ is marginally decomposable.

For future use, we also observe that the unique module of $G(V - \{p\})$ is either $V - \{p, a\}$ or $\{a, u\}$ for some $u \in V - \{p, a\}$, from Lemma 8. Observe that $u \neq b$ because otherwise $\{a, b, p, q\}$ would be a module of G. We have $e_{ub} = e_{ab} \neq e_{qb}$, so $u \neq q$. So $u \in V - \{a, b, p, q\}$.

The case of $w = q$ is similar.

3.2.3. Sub-case $w \in V - \{a, b, p, q\}$

This is the most non-trivial case among all. We begin with some useful results.

Lemma 20. *$\{p, q\}$ is not the module of $G(V - \{a, b, w\})$ for any $w \in V - \{a, b, p, q\}$.*

Proof. Assuming the contrary let $\{p, q\}$ be the module of $G(V - \{a, b, w\})$ for some w. Both p and q cannot be inside X otherwise $\{p, q\}$ would be a module of $G(X)$. Let $p \in V - X$. From the case of $w = p$ above, the unique non-trivial module of $G(V - \{p\})$ is either $V - \{a, p\}$ or $\{a, u\}$ for some $u \in V - \{a, b, p, q\}$. Therefore either $V - \{a, b, p, q\}$ or $\{q, u\}$ is the non-trivial module of $G(V - \{a, b, p\})$, because $\{a, q\}$ is a module in $V - \{b\}$.

Case 1. $\{q, u\}$ is the non-trivial module in $G(V - \{a, b, p\})$:
If $u \neq w$, then $\{p, q, u\}$ is also a module of $G(V - \{a, b, w\})$ which is not possible since $G(V - \{a, b, w\})$ is marginally decomposable. If $u = w$, then $\{p, q, u\}$ is a module of $G(V - \{a, b\})$ because $\{p, q\}$ is assumed to be a module of $G(V - \{a, b, w\})$, which is also not possible as it is an indecomposable graph.

Case 2. $V - \{a, b, p, q\}$ is the non-trivial module in $G(V - \{a, b, p\})$:
In this case q is global to $V - \{a, b, p\}$, in particular q is global to $V - \{a, b, p, w\}$. Combining this with the fact that $\{p, q\}$ is a module of $V - \{a, b, w\}$ we deduce that $V - \{a, b, p, q, w\}$ should be the module of $G(V - \{a, b, w\})$. This is impossible because $a, b, w \notin X$ and $|X| \geq 4$ so $V - \{a, b, p, q, w\}$ is non-trivial and $\{p, q\}$ is another module of marginally decomposable graph $V - \{a, b, w\}$. $\square$

Lemma 21. *Let M be a module in $G(V - \{w\})$ such that $|M \cap \{a, b, p, q\}| > 1$, then $M = V - \{x, w\}$ for some $x \in V - \{a, b, w\}$.*

Proof. First consider the case that $V - \{w, a, b, p, q\} \subseteq M$. Recall that $\{a, q\}$ and $\{b, p\}$ are modules of $V - \{b\}$ and $V - \{a\}$ respectively. If M is $V - \{w, a, b\}$ or $V - \{w, a, p\}$ or $V - \{w, b, q\}$ or $V - \{w, p, q\}$, then $M' = V - \{w, a, b, p, q\}$ will be a module of $V - \{w, a, b\}$ which is not possible for a marginally decomposable graph because of its size. M cannot be $V - \{w, a\}$ or $V - \{w, a, q\}$ since $e_{ba} \neq e_{pa}$. Similarly $V - \{w, b\}$ and $V - \{w, b, p\}$ are not possible values of M. Therefore possible values for M are $V - \{w, p\}$ and $V - \{w, q\}$.

Now consider the second case, i.e., $x \notin M$ for some $x \in V - \{w, a, b, p, q\}$. The cases of $M \cap \{a, b, p, q\} = \{a, q\}$ or $M \cap \{a, b, p, q\} = \{a, q, p\}$ are also not possible because $e_{ab} \neq e_{qb}$. Similarly $M \cap \{a, b, p, q\}$ cannot be $\{b, p\}$ or $\{b, p, q\}$. So M contains some pair of $\{a, q\} \times \{b, p\}$. Then $M' = (M - \{a, b, p, q\}) \cup \{p, q\}$ must be a module of $V - \{a, b, w\}$. This module is non-trivial because x does not belong to it and it has at least 2 elements. From the previous lemma we know that M' cannot be equal to $\{p, q\}$ so M' must contain more than 2 elements. Since $G(V - \{a, b, w\})$ is marginally decomposable, $M' = V - \{a, b, w, x\}$. So $V - \{a, b, p, q, x\} \subseteq M$. Suppose M is $V - \{w, x, a, b\}$ or $V - \{w, x, p, q\}$ or $V - \{w, x, a, p\}$ or $V - \{w, x, b, q\}$, then $M'' = V - \{w, x, a, b, p, q\}$ is also a non-trivial module of $V - \{a, b, w\}$. But this is not possible since $M'' \neq M'$ and $V - \{a, b, w\}$ is marginally decomposable. This leaves three possibilities for M: $V - \{w, x, p\}$, $V - \{w, x, q\}$, and $V - \{w, x\}$. In the first case $V - \{a, b, w, x, p\}$ would be a module of $V - \{a, b, w\}$, which is not possible because M' is another module of $V - \{a, b, w\}$. Similarly $M = V - \{w, x, q\}$ is also not possible. So the only possible value of M is $V - \{w, x\}$. $\square$

We have another result about the structure of the modules of $G(V - \{w\})$.

Lemma 22. *Let M be a module in $G(V - \{w\})$ such that $|M| > 2$, then $M = V - \{x, w\}$ for some $x \in V - \{a, b, w\}$.*

Proof. If $|M \cap \{a, b, p, q\}| > 1$ then we conclude the desired claim from the previous lemma. So assume that $|M \cap \{a, b, p, q\}| \leq 1$. If $|M \cap \{a, b, p, q\}| = 0$ then M is also a module of $V - \{a, b, w\}$, but that is not possible since its size is less than $|V - \{a, b, w\}| - 1$. Next consider the case where $|M \cap \{a, b, p, q\}| = 1$. Let that element be z. Suppose $z = a$. Let x be any element of $M - \{a, b, p, q, w\}$. So $e_{xb} = e_{ab} \neq e_{ap} = e_{xp}$. But this is not possible since $e_{xb} = e_{px}$. Similarly z cannot be b. If $z = p$, then M is also a module of $V - \{a, b, w\}$. Since $q \notin M$, M is non-trivial. By virtue of M being a module, b is global w.r.t. M. So p is global to $M - \{p\}$. Hence $M' = M - \{p\}$ is also a non-trivial module of $V - \{a, b, w\}$, since $|M'| \geq 2$. This is not possible because $V - \{a, b, w\}$ is marginally decomposable. Similarly we can show that $z = q$ is also not possible.
$\square$

The above two results imply that any non-trivial module M of $V - \{w\}$ can be only one of the following: $\{x, y\}$, $\{a, x\}$, $\{q, x\}$, $\{b, x\}$, $\{p, x\}$, $V - \{p, w\}$, $V - \{q, w\}$, and $V - \{w, x\}$, where x, y are arbitrary vertices from the set

Table 1
Module M' of $V - \{a, b, w\}$ corresponding to module M of $V - \{w\}$

$w \in V - \{a, b\}$	M	Range of vars.	M'
Case: $a \in [V - \{a, b\}]$, $b \in eq_{V-\{a,b\}}$			
$w = p$	$\{a, x\}$	$x \in V - \{a, b, p\} - X$	$V - \{a, b, p, x\}$
$w \notin \{a, b, p\}$	$\{x, y\}$	$x, y \in V - \{a, b, w\}$	$\{x, y\}$
	$V - \{w, x\}$	$x \in V - \{a, b, p, w\}$	$V - \{a, b, w, x\}$
Case: $a, b \in eq_{V-\{a,b\}}$			
$w = p$	$\{a, x\}$	$x \in V - \{a, b, p, q\}$	$\{q, x\}$
	$V - \{a, p\}$		$V - \{a, b, p, q\}$
$w = q$	$\{b, x\}$	$x \in V - \{a, b, p, q\}$	$\{p, x\}$
	$V - \{a, q\}$		$V - \{a, b, p, q\}$
$w \notin \{a, b, p, q\}$	$\{x, y\}$	$x, y \in V - \{a, b, w, p, q\}$	$\{x, y\}$
	$\{x, p\}$ or $\{x, b\}$	$x \in V - \{a, b, w, p, q\}$	$\{x, p\}$
	$\{x, q\}$ or $\{x, a\}$	$x \in V - \{a, b, w, p, q\}$	$\{x, q\}$
	$V - \{w, x\}$	$x \in V - \{a, b, w\}$	$V - \{a, b, w, x\}$

$V - \{a, b, p, q, w\}$. In the respective cases the unique module of $V - \{a, b, w\}$ will be $\{x, y\}$, $\{q, x\}$, $\{q, x\}$, $\{p, x\}$, $\{p, x\}$, $V - \{a, b, p, w\}$, $V - \{a, b, q, w\}$, and $V - \{a, b, x, w\}$. This indicates that $V - \{w\}$ can have more than one module in two cases: (i) when $\{q, x\}$ is a module of $V - \{a, b, w\}$ then $\{a, x\}$ and $\{q, x\}$, both, may be modules of $V - \{w\}$; (ii) when $\{p, x\}$ is a module of $V - \{a, b, w\}$, there may be two modules of $V - \{w\}$, namely, $\{b, x\}$ and $\{p, x\}$. If $\{a, x\}$ and $\{q, x\}$ both are modules of $V - \{w\}$, then $e_{xb} = e_{ab} \neq e_{qb} = e_{xb}$. So this is not possible. Similarly the second case is not possible. Thus we conclude that $V - \{w\}$ has at most one module and its size is 2 or $|V| - 2$.

Combining the three sub-cases we have the following lemma.

Lemma 23. *Given that $G = (V, E)$ is X-critical. If (a, b) is a locked pair with a and b both in $eq_{V-\{a,b\}}$ and if $G(V - \{a, b\})$ is X-stable, then G is also X-stable.*

The main result of this section follows.

3.3. Main theorem

From Observation 3 and Lemmas 11, 19 and 23, and the fact that $G(X)$ is vacuously X-stable if it is indecomposable, we deduce the following theorem using induction.

Theorem 24. *Every X-critical graph is X-stable.*

In this section we have shown that if an X-critical graph G has a locked pair (a, b) and $G(V - \{a, b\})$ is X-stable, then G is also X-stable. The proof explicitly constructs the unique module of $V - \{w\}$ for each $w \notin X$. In Table 1 we summarize the module M of $G(V - \{w\})$ and the module M' of $G(V - \{a, b, w\})$ for $w \notin \{a, b\}$ for various cases, which will be useful in Section 5.

4. A commutative elimination sequence

Proposition 25. *Let G be a X-critical graph, then (a, b) is a locked pair of G iff $a, b \in V - X$ and the unique non-trivial module of $G(V - \{b\})$ is $V - \{a, b\}$ or $\{a, q\}$ for some $q \in V - \{a, b\}$.*

Proof. (if) Suppose M is a non-trivial module of $V - \{a, b\}$. Based on Lemma 8 we consider two cases. (i) Let $\{a, q\}$ be a module of $V - \{b\}$. If $q \in M$, then $M \cup \{a\}$ is another module of $V - \{b\}$. If $q \notin M$, then M is another module of $V - \{b\}$. (ii) Next let $V - \{a, b\}$ be a module of $V - \{b\}$. Then M is also a module of $V - \{b\}$ because a is global to $V - \{a, b\}$. So uniqueness of the module of $V - \{b\}$ requires that $V - \{a, b\}$ is indecomposable.

(only if) From Lemma 8. $\square$

Lemma 26. *Let (a, b) and (c, d) be locked pairs in an X-critical graph G, with no common vertex. Then (a, b) is also a locked pair in $G(V - \{c, d\})$.*

Proof. From Lemma 9 and Theorem 24, $V - \{b, c, d\}$ is marginally decomposable with the unique module M. At least one of a and b is in $eq_{V-\{a,b\}}$. Without loss of generality assume that $\{a, q\}$ is the module of $G(V - \{b\})$.

(i) Case of $q \notin \{c, d\}$: In this case $\{a, q\}$ is also a module of $V - \{b, c, d\}$ so $M = \{a, q\}$. So from Proposition 25 $\{a, b\}$ is a locked pair of $V - \{c, d\}$.

(ii) Case of $q \in \{c, d\}$: Without loss of generality let $q = c$. From Proposition 25, either $V - \{c, d\}$ or $\{c, e\}$ is the module of $V - \{d\}$.

(a) In the former case c is global to $V-\{c,d\}$ so a is global to $V-\{a,b,c,d\}$. Thus $M=V-\{a,b,c,d\}$. Then from Proposition 25, $\{a,b\}$ is a locked pair of $V-\{c,d\}$.

(b) Next consider the latter case. $\{a,c\}$ is the module in $G(V-\{b\})$ and $\{c,e\}$ is the module in $G(V-\{d\})$. If $e=a$ then $\{a,c\}$ will be a module of G which is not possible. If $e=b$ then $\{a,b,c\}$ will be a module of $G(V-\{d\})$ which is not possible, due to the size, since $G(V-\{d\})$ is marginally decomposable. Thus we find that $e\in V-\{a,b,c,d\}$. Then $\{a,e\}$ is the module of $G(V-\{b,c,d\})$. Once again from Proposition 25 (a,b) is a locked pair in $G(V-\{c,d\})$. □

From Corollary 12 we know that in an X-critical graph $V-X$ has even number of vertices. Let $V=X\cup\{a_1,b_1,\ldots,a_k,b_k\}$. Then set $\{(a_1,b_1),\ldots,(a_k,b_k)\}$ is called a *X-criticality preserving commutative elimination sequence* if $G(V-\{a_{j_1},b_{j_1},\ldots,a_{j_i},b_{j_i}\})$ is X-critical for any subset $\{j_1,j_2,\ldots,j_i\}$ of $\{1,\ldots,k\}$.

Corollary 27. *Let G be an X-critical graph with $V=X\cup\{a_1,b_1,\ldots,a_k,b_k\}$. Then $\{(a_1,b_1),\ldots,(a_k,b_k)\}$ is a commutative elimination sequence iff (a_j,b_j) is a locked pair of G for all j.*

Lemma 28. *Let G be X-critical and (a,b) be a locked pair in it. If $G(V-\{a,b\})$ has a commutative elimination sequence, then so does G.*

Proof. Suppose ES' is a commutative elimination sequence of $G(V-\{a,b\})$. We will first show that at most one locked pair in ES' may not remain a locked pair of G.

Consider a locked pair $(c,d)\in ES'$ in which at least one vertex w is such that neither $\{w,a\}$ is a module in $G(V-\{b\})$ nor $\{w,b\}$ is a module in $G(V-\{a\})$. So w is neither p nor q in the sense of the previous section. Let v be the other vertex of $\{c,d\}$. The module M' of $G(V-\{a,b,w\})$ must be either $V-\{a,b,w,v\}$ or $\{v,x\}$ for some x because $G(V-\{a,b,w\})$ is marginally decomposable. From Table 1 (rows of $w\notin\{a,b,p\}$ and $w\notin\{a,b,p,q\}$) we find that the module M of $G(V-\{w\})$ is unique and it is given by: $M=M'$ if $M'=\{v,x\}$; $M=V-\{w,v\}$ when $M'=V-\{a,b,w,v\}$. Thus M is either $V-\{w,v\}$ or $\{v,x\}$ for some $x\in V-\{a,b,w,v\}$. From Proposition 25 we deduce that (w,v) (which is same as (c,d)) remains a locked pair in G. If all pairs on E' are found to remain locked pairs in G, then $ES=ES'\cup\{(a,b)\}$ is a commutative elimination sequence of G.

In case not all locked pairs of $G(V-\{a,b\})$ are locked pairs of G then the exception must be only one pair (p,q) where $\{a,q\}$ is the module of $G(V-\{b\})$ and $\{b,p\}$ is the module of $G(V-\{a\})$. From Table 1 (second entry for $w=p$), if $\{q,u\}$ is the module of $G(V-\{a,b,p\})$, then $\{a,u\}$ is the unique module of $G(V-\{p\})$. Thus from Proposition 25 (a,p) is a locked pair of G. Otherwise if $V-\{a,b,p,q\}$ is the module of $V-\{a,b,p\}$, then $V-\{a,p\}$ is the unique module of $V-\{p\}$. From Proposition 25, $\{a,p\}$ is a locked pair of G. A similar argument shows that (b,q) is also a locked pair of G. In this case $E=E'-\{(p,q)\}\cup\{(a,p),(b,q)\}$ is a commutative elimination sequence. □

Vacuously, the subgraph $G(X)$ of X-critical G has a commutative elimination sequence. By induction and using Lemma 28 we have a trivial conclusion that every X-critical graph has a commutative elimination sequence.

Theorem 29. *Every X-critical graph has a commutative elimination sequence.*

Theorem 30. *The commutative elimination sequence in an X-critical graph is unique.*

Proof. Let ES_1 and ES_2 be two distinct commutative elimination sequences. Assume $\{a_1,b_1\}$ is a locked pair in ES_1 which is not a locked pair in ES_2 then there must be locked pairs $\{a_1,b_2\}$ and $\{a_2,b_1\}$ in ES_2.

If $V-\{b_1\}$ has the module M of size $|V-\{b_1\}|-1$, then that module should be $V-\{a_1,b_1\}$ since (a_1,b_1) is a locked pair in G. But (a_2,b_1) is also a locked pair so the module should be $V-\{a_2,b_1\}$, implying that $a_1=a_2$ which is not true. So the module size must be 2. Since (a_1,b_1) is a locked pair so $M=\{a_1,x\}$. Similarly (a_2,b_1) is also a locked pair so $M=(a_2,y)$. Thus $M=(a_1,a_2)$. Similarly $V-\{b_2\}$ also has (a_1,a_2) as its unique module. Together these assertions imply that (a_1,a_2) is a module of entire G, which is absurd. □

4.1. *Computing elimination sequences in X-critical graphs*

We give here a method to calculate a commutative elimination sequence. This algorithm is similar to the $O(n+m\log n)$ algorithm by Cournier and Habib [13] for the computation of maximal modular decomposition.

Let $G=(V,E)$ be an X-critical graph and x,y be a pair of vertices in $V-X$ such that $G(V-\{x,y\})$ is also X-critical, then $\{x,y\}$ is called a *locked pair* in G. We have observed that at least one of the vertices of the pair is from class $eq_{V-\{x,y\}}$. The other will be either from $eq_{V-\{x,y\}}$ or from $[V-\{x,y\}]$.

Algorithm 1 computes the commutative elimination sequence of an X-critical graph. Starting from $Y=X$ we expand Y to V identifying one locked pair in each step. The basic technique is based on computing $\mathcal{C}(V-Y,Y)$, see paragraph following Corollary 12. If $\mathcal{C}$ denotes $\mathcal{C}(V-Y,Y)$ and $x\in V-Y$, then *update*$(\mathcal{C},a)$ computes $\mathcal{C}(V-Y-\{a\},Y\cup\{a\})$ from $\mathcal{C}(V-Y,Y)$.

Algorithm 1. Computation of elimination sequence.

Data: X-critical graph G, set X
Result: Commutative elimination sequence
1. $\mathcal{C} = \mathcal{C}(V - X, X)$;
$extn(X)$ **will be empty**
3. $Y = X$;
for $i = 1$ *to* $k = (|V| - |X|)/2$ **do**
4. $\mathcal{C}' = \mathcal{C}$;
5. Select any vertex a_i from some class $eq(u)$ of $\mathcal{C}$;
6. $\mathcal{C} = update(\mathcal{C}, a_i)$;
7. $b_i =$ An arbitrary vertex from $extn(Y \cup \{a_i\})$;
8. $\mathcal{C} = update(\mathcal{C}, b_i)$;
9. **If** $a_i \in eq(u)$ *and* $b_i \in eq(v)$ *in* $\mathcal{C}'$ *and* $(u, v) = (a_j, b_j)$ *for some* $j < i$
then $(a_j, b_j) = (a_i, v)$ and $(a_i, b_i) = (u, b_i)$;
10. $Y = Y \cup \{a_i, b_i\}$
11. **return** $(a_1, b_1), \ldots, (a_k, b_k)$

Step 5 is based on the fact that at least one vertex of every locked pair is from class $eq()$. Step 9 ensures that if $(a_1, b_1), \ldots, (a_{i-1}, b_{i-1})$ is the commutative elimination of $G(Y_{i-1})$, then $(a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_i, b_i)$ is the elimination sequence of $G(Y_i)$, which is based on the proof of Lemma 28.

The update steps take $O(n)$ time. The first step takes $O(n^2 + m)$ time since for each $x \in V - X$ it needs to be found out if X is a module of $G(X \cup \{x\})$ or if there there is $u \in X$ such that $\{x, u\}$ is a module of $G(X \cup \{x\})$ or not. Therefore the entire process costs $O(n^2)$.

Theorem 31. *The commutative elimination sequence for an X-critical graph can be computed in $O(n^2)$.*

4.2. Ille's theorem: An alternate proof

In this section we shall show that if an indecomposable graph $G = (V, E)$ has an indecomposable subgraph $G(X)$ with $|V - X| > 5$ then a pair of vertices $a, b \in V - X$ can be computed in $O(n(n + m))$ time such that $G(V - \{a, b\})$ is also indecomposable.

To find a pair of vertices from $V - X$ such that the reduced graph after deleting the pair remains indecomposable, we may randomly delete a vertex and test the resulting graph for indecomposability. If this test fails for every vertex in $V - X$, then the graph is X-critical and we have already seen how to find a locked pair. If it succeeds for some vertex a, then we repeat this step on $G(V - \{a\})$. If this succeeds again, then we have the desired pair. The difficult case is when after deleting one vertex the graph reduces to X-critical. The following result addresses the problem of locating such a pair in these graphs.

Theorem 32. *Let G be an indecomposable graph on (V, E), X be a subset of V and $V - X = \{a, a_1, b_1, a_2, b_2, a_3, b_3\}$ where $G(V - \{a\})$ is X-critical and $p_1 = \{a_1, b_1\}$, $p_2 = \{a_2, b_2\}$, $p_3 = \{a_3, b_3\}$ is a commutative elimination sequence in $G(V - \{a\})$. Then for at least one locked pair, $p_i = \{a_i, b_i\}$, $G(V - \{a_i, b_i\})$ is indecomposable.*

Proof. Assume the contrary. Denote $V - p_i$ by Z_i. From the assumption $G(Z_i)$ is decomposable but $G(V - \{a\} - p_i)$ is indecomposable (actually X-critical) from the definition of commutative elimination sequence. It is known from Lemma 8 that if a subgraph $G(A)$ is indecomposable and $G(A \cup \{a\})$ is decomposable, then the latter has a unique module and it is either A or $\{a, b\}$ for some $b \in A$. Therefore either $a \in [Z_i - \{a\}]$ or $a \in eq_{Z_i-\{a\}}(u_i)$ for each i where u_i is some vertex in $Z_i - \{a\}$.

Assume that $a \in [Z_1 - \{a\}]$ and $a \in [Z_2 - \{a\}]$. Since $(Z_1 - \{a\}) \cap (Z_2 - \{a\})$ is non-empty, $(Z_1 - \{a\}) \cup (Z_2 - \{a\}) = V - \{a\}$ is a module of G which is absurd as G is indecomposable. Therefore $a \in [Z_i - \{a\}]$ for not more than one i. Without loss of generality, either a belongs to $[Z_1 - \{a\}]$, $eq_{Z_2-\{a\}}(u_2)$, and $eq_{Z_3-\{a\}}(u_3)$; or a belongs to $eq_{Z_i-\{a\}}(u_i)$ for all i. In the following discussions we show that these possibilities also lead to conflicts.

If $u_j = u_k = u$ for some $j \neq k$, then $\{a, u\}$ is a module of G, which is not possible as G is indecomposable. Thus $u_j \neq u_k$ for $j \neq k$.

Further if u_j and u_k both belong to $V - p_j - p_k$, then $\{a, u_j\}$ and $\{a, u_k\}$ are both modules in $G(V - p_2 - p_3)$ therefore $\{u_j, u_k\}$ must be a module in $G(V - \{a\} - p_j - p_k)$. This is impossible since the definition of commutative elimination sequence requires that $G(V - \{a\} - p_j - p_k)$ is X-critical. So we conclude that either $u_j \in p_k$ or $u_k \in p_j$. These observations lead to only two possibilities.

Case 1: Assume that $V - p_1 - \{a\}$, $\{a, u_2\}$ and $\{a, u_3\}$ are the modules of $G(Z_1)$, $G(Z_2)$ and $G(Z_3)$ respectively. From the previous paragraph we know that $u_3 \in p_2$ or $u_2 \in p_3$. Without loss of generality assume the latter. The facts that $V - p_1 - \{a\}$ is a module in $G(V - p_1)$ and $\{a, u_2\}$ is a module in $G(V - p_2)$ imply that $V - p_1 - p_2 - \{a, u_2\}$ is a module in $G(V - p_1 - p_2 - \{a\})$. This is absurd because $G(V - p_1 - p_2 - \{a\})$ is X-critical.

Case 2: Assume that $\{a, u_i\}$ is the module in $G(V-p_i)$ for all i. From the earlier observation all u_i are distinct and the following are true:

(i) $u_1 \in p_2$ or $u_2 \in p_1$,
(ii) $u_2 \in p_3$ or $u_3 \in p_2$, and
(iii) $u_3 \in p_1$ or $u_1 \in p_3$.

These condition require that $u_1 \in p_2, u_2 \in p_3, u_3 \in p_1$ or $u_1 \in p_3, u_2 \in p_1, u_3 \in p_2$. Without loss of generality assume the first with $u_1 = a_2, u_2 = a_3, u_3 = a_1$, as there is nothing here to distinguish between a_i and b_i.

Here, $\{a, u_1\} = \{a, a_2\}$ is a module in $G(V - p_1)$ and $\{a, a_3\}$ is a module in $G(V - p_2)$. Combining the two we have $\{a, a_2, a_3\}$ is a module in $G(V - p_1 - \{b_2\})$. Similarly $\{a, a_3, a_1\}$ is a module in $G(V - p_2 - \{b_3\})$ and $\{a, a_1, a_2\}$ is a module in $G(V - p_3 - \{b_1\})$. Together they imply that $\{a, a_1, a_2, a_3\}$ is a module in $G(V - \{b_1, b_2, b_3\})$. We can derive another fact from these three modules. $\{a, a_2, a_3\}$ is a module in $G(V - p_1 - \{b_2\})$ so $\{a_2, a_3\}$ is a module in $G(V - \{a\} - p_1 - \{b_2\})$. While $\{a_2, a_3\}$ cannot be a module of $G(V - \{a\} - p_1)$ because the latter is X-critical, it is necessary that $e_{a_2b_2} \neq e_{a_3b_2}$. Since $\{a, a_1, a_2\}$ is a module in $G(V - p_3 - \{b_1\})$, $e_{ab_2} = e_{a_1b_2} = e_{a_2b_2}$. These relations and similar other relations are stated below:

$$\begin{aligned} &\text{(i) } e_{ab_2} = e_{a_1b_2} = e_{a_2b_2} \neq e_{a_3b_2} \\ &\text{(ii) } e_{ab_1} = e_{a_3b_1} = e_{a_1b_1} \neq e_{a_2b_1} \\ &\text{(iii) } e_{ab_3} = e_{a_2b_3} = e_{a_3b_3} \neq e_{a_1b_3}. \end{aligned} \tag{1}$$

As $\{a_1, b_1\}$ is a locked pair in $G(V - \{a\})$, either $a_1 \in [V - \{a\} - p_1]$ or $\{a_1, v_1\}$ is a module in $G(V - \{a, b_1\})$ for some $v_1 \in V - \{a, a_1, b_1\}$. Assume the former, i.e., $a_1 \in [V - \{a\} - p_1]$. We know that $\{a, a_1, a_2\}$ is a module in $G(V - p_3 - \{b_1\})$ so a_2 must be in $[V - p_1 - p_3 - \{a, a_2\}]$. This implies that $G(V - \{a\} - p_1 - p_3)$ is decomposable which is not true as it is X-critical. So $\{a_1, v_1\}$ must be the module in $G(V - \{a, b_1\})$. Similarly there exist v_2, v_3 such that $\{a_2, v_2\}$ is the module in $G(V - \{a, b_2\})$ and $\{a_3, v_3\}$ is the module in $G(V - \{a, b_3\})$.

Next we will show that v_i is b_j for some $j \neq i$. Firstly, $\{a_1, v_1\}$ is a module of $G(V - \{a, b_1\})$ so $e_{a_1b_2} = e_{v_1b_2}$. From relations (1) we find that $v_1 \neq a_3$. Similarly $e_{a_1b_3} = e_{v_1b_3}$ implies that $v_1 \neq a_2$. Similar arguments establish that $\{v_1, v_2, v_3\} \cap \{a_1, a_2, a_3\} = \emptyset$. Secondly, suppose $v_1 \in V - p_1 - p_2 - p_3 - \{a\}$. Using the fact that $\{a, a_1, a_2\}$ is a module of $G(V-p_3-\{b_1\})$ we can deduce that $\{v_1, a_2\}$ is a module of $G(V-\{a\}-p_1-p_3)$ which is not possible for an X-critical graph. As $\{a_1, v_1\}$ is a module in $V - \{a, b_1\}$, $v_1 \neq b_1$. Thus we find that $v_1 \in \{b_2, b_3\}$. Similarly $v_2 \in \{b_3, b_1\}$ and $v_3 \in \{b_1, b_2\}$. We further show that all v_i are distinct. Let $v_1 = v_3 = b_2$. Now $\{a_1, v_1\}$ is a module in $G(V - \{a, b_1\})$ so $e_{a_3a_1} = e_{a_3b_2}$. Also, $\{a_3, v_3\}$ is a module of $G(V - \{a, b_3\})$ so $e_{a_1a_3} = e_{a_1b_2}$. This means $e_{a_1b_2} - e_{a_3,b_2}$, which contradicts the first of relations 1. Thus $\{v_1, v_2, v_3\} = \{b_1, b_2, b_3\}$.

Finally we put together the facts that $\{a_i, v_i\}$ is a module of $G(V-\{a, b_i\})$, $\{a, a_1, a_2, a_3\}$ is a module of $G(V-\{b_1, b_2, b_3\})$, and $\{v_1, v_2, v_3\} = \{b_1, b_2, b_3\}$. Consequently $\{a, a_1, a_2, a_3, b_1, b_2, b_3\}$ is a module of G, which is absurd as we had started with the assumption that G is indecomposable. So Case 2 is also impossible. □

Corollary 33. *Let G be an indecomposable graph with $a \in V$ such that $G(V-\{a\})$ is X-critical and $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ is its commutative elimination sequence. Then out of any three locked pairs of the sequence, there exists at least one pair (a_j, b_j) such that $G(V - \{a_j.b_j\})$ is indecomposable.*

Corollary 34. *Let $G = (V, E)$ be an indecomposable graph containing an indecomposable subgraph $G(X)$ and $|V-X| > 5$. Then a pair of vertices $a, b \in V - X$ can be computed in $O(n(n + m))$ time such that $G(V - \{a, b\})$ is also indecomposable.*

Proof. For each vertex $a \in V - X$ check if $G(V - \{a\})$ is indecomposable until one such vertex is located. (i) If no such vertex exists, then G is X-critical and from Theorem 31 we can compute a complete elimination sequence in $O(n^2)$ time. So total cost of the computation is $O(n(n + m) + n^2)$ because indecomposability can be tested in $(n + m)$. If a vertex a is located, then locate a vertex b in $V - X - \{a\}$ such that $G(V - \{a, b\})$ is indecomposable. (ii) If one such vertex is located such that $G(V - \{a\})$ is indecomposable, then a, b is the desired pair and the cost of the computation is $O(n(n + m))$. (iii) Otherwise $G(V - \{a\})$ is X-critical. Since $|V-X-\{a\}| > 4$, there are at least three locked pairs in the elimination sequence of $G(V - \{a\})$. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are any three pairs in the sequence. From the previous corollary we know that at least one of these pairs can be removed from G while preserving indecomposability. Therefore we compute the commutative elimination sequence of $G(V - \{a\})$ in $O(n^2)$ time and check the indecomposability of $G(V - \{a_i, b_i\})$, for each $i = 1, 2, 3$. Then the desired pair is a_i, b_i if $G(V - \{a_i, b_i\})$ is indecomposable. The testing of indecomposability of the three subgraphs costs $O(n + m)$, so total cost in this case is $O(n(n + m) + n^2) = O(n(n + m))$. □

An obvious consequence of this result is that an indecomposability preserving elimination sequence can be computed in $O(n^2(n + m))$.

Corollary 35. *Let $G = (V, E)$ be an indecomposable graph containing an indecomposable subgraph $G(X)$ with $|V - X| > 5$. Then a sequence $D_1, D_2, \ldots, D_k$ of vertex pairs can be computed in $O(n^2(n + m))$ such that these pairs are mutually exclusive, $D_i \subset V - X$, $G(V - D_1 - D_2 \cdots - D_i)$ is indecomposable for $1 \leq i \leq k$, and $|V - X - D_1 - \cdots - D_k| \leq 5$.*

Table 2
Necessary and sufficient test for $G(Y)$ to be X-critical

w	M'	M		Condition
Case: $a \in [Y']$, $b \in eq_{Y'}(p)$				
$w = p$	$Y' - \{p, x\}$	$\{a, x\}$	$x \in Y' - \{p\}$	$e_{ab} = e_{xb}$
$w \notin \{a, b, p\}$	$\{x, y\}$	$\{x, y\}$	$x, y \in Y' - \{w\}$	$e_{vb} = e_{xb}$
	$Y' - \{w, x\}$	$Y - \{w, x\}$	$x \in Y' - \{p, w\}$	$e_{aw} \neq e_{vw}$
Case: $a \in eq_{Y'}(q)$, $b \in eq_{Y'}(p)$				
$w = p$	$\{q, x\}$	$\{a, x\}$	$x \in Y' - \{p, q, w\}$	$e_{aq} = e_{xq}$
	$Y' - \{p, q\}$	$Y - \{a, p\}$		$e_{aq} = e_{ab}$
$w = q$	$\{p, x\}$	$\{b, x\}$	$x \in Y' - \{p, q, w\}$	$e_{bp} = e_{xp}$
	$Y' - \{p, q\}$	$Y - \{b, q\}$		$e_{bp} = e_{ba}$
$w \notin \{a, b, p, q\}$	$\{x, y\}$	$\{x, y\}$	$x, y \in Y' - \{w, p, q\}$	*true*
	$\{x, p\}$	$\{x, b\}$	$x \in Y' - \{w, p, q\}$	$e_{bx} = e_{bp}$ $\& e_{ax} = e_{ab}$
	$\{x, p\}$	$\{x, p\}$	$x \in Y' - \{w, p, q\}$	$e_{bx} = e_{bp}$ $\& e_{ax} \neq e_{ab}$
	$\{x, q\}$	$\{x, a\}$	$x \in Y' - \{w, p, q\}$	$e_{ax} = e_{aq}$ $\& e_{bx} = e_{ba}$
	$\{x, q\}$	$\{x, q\}$	$x \in Y' - \{w, p, q\}$	$e_{ax} = e_{aq}$ $\& e_{bx} \neq e_{ba}$
	$Y' - \{w, x\}$	$Y - \{w, x\}$	$x \in Y' - \{w, p, q\}$	*true*
	$Y' - \{w, p\}$	$Y - \{w, p\}$		$e_{bp} = e_{qp}$
	$Y' - \{w, q\}$	$Y - \{w, q\}$		$e_{aq} = e_{pq}$

5. Maximal X-critical subgraph

Let $G = (V, E)$ be an arbitrary graph which has an indecomposable subgraph $G(X)$. In this section we will discuss the computation of a maximal X-critical subgraph of G.

Let $G(Y')$ be an X-critical subgraph and $a, b \in V - Y'$ such that $G(Y)$ be indecomposable where $Y = Y' \cup \{a, b\}$. In addition, either $a \in [Y']$ and $b \in eq_{Y'}$ or both a, b belong to $eq_{Y'}$, satisfying the condition: (i) if $a \in [Y']$, then $e_{ab} \neq e_{ap}$ (ii) if $a \in eq_{Y'}(q)$, then $e_{ap} = e_{bq} \neq e_{ab}$, where $b \in eq_{Y'}(p)$. Then graph $G(Y)$ is X-critical iff $G(Y - \{w\})$ is marginally decomposable for all $w \in Y - X$. Due to the choice of a, b it is sufficient to test the condition for $w \in Y' - X$.

Table 1 lists the module M' of $G(Y' - \{w\})$ corresponding to the module M of $G(Y - \{w\})$. To check that $G(Y - \{w\})$ is marginally decomposable we must ensure that M given in the table is its unique module. Table 2 lists the necessary and sufficient conditions for the same. The conditions given in the table establishes the existence and uniqueness of M as the module. The conditions are derived using the fact that $M \cap Y'$ is either M' or a trivial module.

Algorithm 2. Computing a maximal X-critical subgraph.

Data: Arbitrary graph $G = (V, E)$ with indecomposable subgraph $G(X)$
Result: A maximal X-critical subgraph of G
$Y' = X$;
$W = V$;
$\mathcal{C}' = \mathcal{C}(W - X, X)$;
initialize $\mathcal{M}[w]\ \forall w \in V - X$ to be empty;
While $eq_{\mathcal{C}'}$ *is non-empty* **do**
select any vertex b from some class $eq_{\mathcal{C}'}(p)$ for some $p \in Y'$;
$\mathcal{C} = update(\mathcal{C}', b)$;
$foundpair = false$;
Repeat
Select next $a \in extn_{\mathcal{C}} - extn_{\mathcal{C}'}$;
Perform the test of Table 1 for $\{a, b\}$;
if *test succeeds* **then**
$\mathcal{C}' = update(\mathcal{C}, a)$;
$Y' = Y' \cup \{a, b\}$;
$W = W - \{a, b\}$;
$update\mathcal{M}[w]\ \forall w \in Y - X$;
$foundpair = true$;
Until *foundpair* or $extn_{\mathcal{C}} - extn_{\mathcal{C}'}$ is exhausted;
If ($foundpair = false$) $\{W = W - \{b\}; \mathcal{C}' = \mathcal{C}' - b\}$;
return Y';

Algorithm 2 computes a maximal X-critical subgraph of an arbitrary graph G which contains an indecomposable subgraph $G(X)$. The algorithm starts with the subgraph $G(Y') = G(X)$ which is vacuously X-critical. Initially we compute $\mathcal{C}(V - X, X)$ and initialize $\mathcal{M}$ to empty array, where $\mathcal{M}[w]$ will store the unique module of $Y' - \{w\}$ for each $w \in Y' - X$. In each cycle we locate a new pair of vertices to be appended to Y and stop when no addition is possible.

If the current Y' is not a maximal X-critical subgraph of G, then there must be a maximal X-critical $G(V')$ which contains Y. So the vertices of $V' - Y'$ should form a commutative elimination sequence, say $\{a_1, b_1\}, \{a_2, b_2\}, \ldots$. Each $\{a_i, b_i\}$ is a locked pair in $G(Y' \cup \{a_i, b_i\})$. Without loss of generality assume that b_i is from class $eq_{Y'}$ since at least one must be from these classes. Since $G(Y' \cup \{a_i, b_i\})$ is X-critical, a_i must belong to $extn(Y' \cup \{b_i\})$.

We start a pass of while-loop with an arbitrarily selected candidate for b_i. In for-loop a compatible a_i is searched. If no suitable a_i is found, then there does not exist any X-critical extension of Y containing b_i so it is deleted from future consideration.

Since new $\mathcal{C}'$ computation and the test of Table 2 are performed in $O(n^2)$ time and each step takes $O(n)$ time, we have the following result.

Theorem 36. *Algorithm 2 computes a maximal X-critical subgraph of a given graph in $O(n^3)$ time.*

6. Two algorithmic problems on X-critical graphs

6.1. Perfect matching

Theorem 37. *Let G be an X-critical graph. If X has a perfect matching then so has G. Given the matching on X a matching on G can be computed in $O(n + m \log n)$.*

Proof. We prove the existence of the matching by induction on $|V|$. If $V = X$ then there is nothing to show. Let us assume that $G(Z)$ is a X-critical subgraph having a perfect matching M. Now we extend this graph by two vertices s.t. $G(Z \cup \{a, b\})$ is also X-critical. We will show that $G(Z \cup \{a, b\})$ has also got a perfect matching M'.

If $(a, b) \in E$ then we can easily extend the matching to include these new vertices $M' = M \cup \{(a, b)\}$. If $(a, b) \notin E$ then we need to consider two cases depending upon the kind of vertices a and b are (Observation 3).

Case 1. $a \in [Z]$, $b \in eq_Z(p)$. As $(a, b) \notin E$, $(a, z) \in E$, $\forall z \in Z$. Let $(p, c) \in M$ (p is matched to some c in M). Then the new matching is given by $M' = M - \{(p, c)\} \cup \{(a, p), (b, c)\}$.

Case 2. $a \in eq_Z(q)$, $b \in eq_Z(p)$. As $(a, b) \notin E$, $(a, p), (b, q) \in E$ and hence $(p, q) \in E$. If $(p, q) \in M$ then a new matching can be given by $M' = M - \{p, q\} \cup \{(a, p), (b, q)\}$. Else let $(q, c), (p, d) \in M$ then $(a, c), (b, d) \in E$ and the new matching is given by $M' = M - \{(p, d), (q, c)\} \cup \{(p, q), (a, c), (b, d)\}$.

A matching on X can be extended to a matching on G in the same way. The dominating factor in the time complexity of this problem is due to the computation for finding the new pairs which we are going to add. As calculation of an elimination sequence has complexity $O(n + m \ln n)$, we can extend the matching in the same complexity (we are doing constant amount of work while extending the matching if the elimination sequence is given). $\square$

Remark 38. The converse of the theorem is not true. For example the graph in Fig. 1 has a perfect matching, while $G(X)$ does not.

6.2. 3-coloring

A odd subdivision of a graph G is a graph that results from G on replacing each of its edges by any path of odd length. Fig. 2 is an example of odd subdivision of K_4.

Toft [14] had conjectured in 1974 that every graph which is free from any odd subdivision of K_4 as its subgraph, is 3-colorable. Zang [15] had proved this conjecture. In this section we give an $O(n(n + m))$ algorithm to 3-color an X-critical graph which is free from odd subdivisions of K_4, if a 3-coloring of $G(X)$ is given.

We begin with a simple but very useful result.

Lemma 39. *Let $G = (V, E)$ be a graph and $x \in V$.*

(i) *G is 3-colorable iff each of the strongly connected components (blocks) of G is 3-colorable.*
(ii) *Let G be 3-colorable. Let $(T_1, E_1), \ldots, (T_s, E_s)$ be the strongly connected components (blocks) of G which share vertex x. Let $F' : \cup_{i=1}^{s} T_i \to \{1, 2, 3\}$ be a 3-coloring for these components. Then G has a 3-coloring $F : V \to \{1, 2, 3\}$ such that $F(y) = F'(y)$ for all $y \in \cup_{i=1}^{s} T_i$.*

The claim is obvious once we observe that any two strongly connected components share at most one vertex which is a cut vertex of the graph.

Theorem 40. *Let G be an X-critical graph and a 3-coloring of $G(X)$ be given. Then in $O(n(n + m))$ time either a 3-coloring of G can be computed or a subgraph of G can be computed which is an odd subdivision of K_4.*

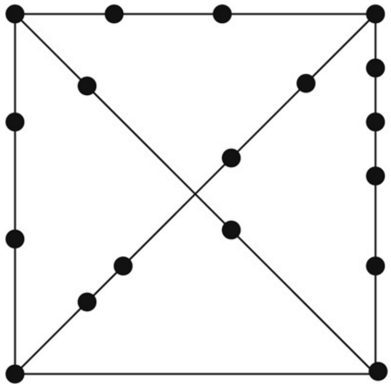

Fig. 2. An example of odd subdivision of K_4.

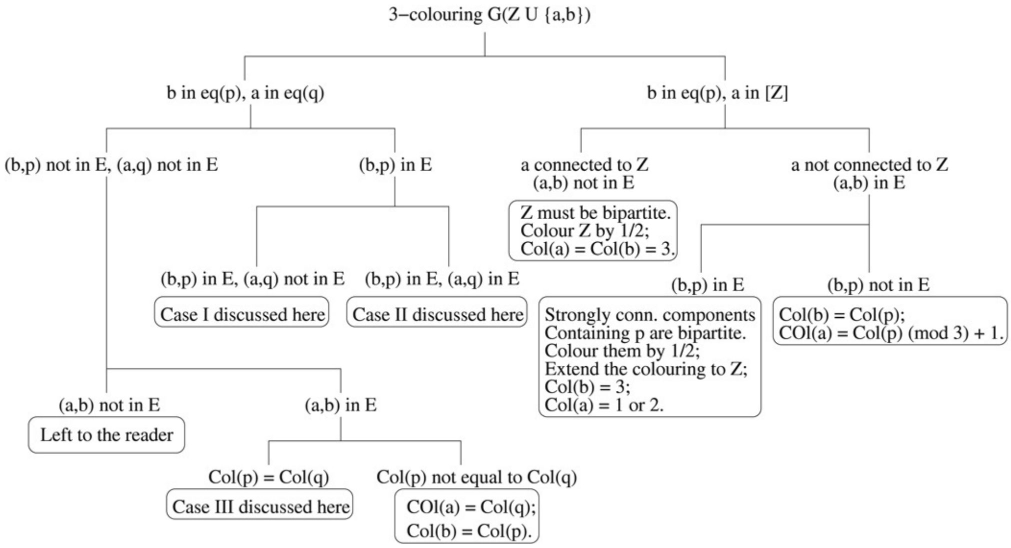

Fig. 3. Case-Tree for 3-coloring proof.

Proof. As in Theorem 37 we will present an iterative algorithm. Starting with $Z = X$ and a locked pair (a, b) of commutative elimination sequence of G, we will show that a 3-coloring of $G(Z \cup \{a, b\})$ can be computed or show that $G(Z \cup \{a, b\})$ contains an odd subdivision of K_4 as its subgraph.

The proof requires the consideration of a large number of cases. For simplifying the structure of the proof, Fig. 3 shows the case-tree. The three cases which need discussion follow.

(I) Case of $(b, p) \in E$ and $(a, q) \notin E$.

Assume $G(Z \cup \{a, b\})$ does not contain any odd subdivision of K_4. This means that p belongs to no odd cycles in $G(Z)$. Consider strongly connected components containing p, we have two different cases. Either q belongs to them or it does not.

Let q belong to a strongly connected component of p. As all strongly connected components containing p are bipartite, these can be colored by any two colors, say 1, 2. Extend the coloring to entire Z, see Lemma 39. This leaves color 3 for b. We color a by the same color as q.

Let q not belong to the strongly connected component of p. Again we color the strongly connected components containing p by 1,2. Also, we color b by 3. If the color of q is not 3 then we color a by the same color.

Otherwise, let x be the cut vertex in the connected components of p which separates p from q. Without loss of generality assume that x is colored by 1. Switch the colors 2 and 3 in the component of $G(V - \{x\})$ containing q. Now q has color 2. Set the color of a to be 2.

(II) Case of $(b, p) \in E$ and $(a, q) \in E$.

Assume $G(Z \cup \{a, b\})$ does not contain any odd subdivision of K_4. This means that neither p nor q belongs to any odd cycles in $G(Z)$. Consider strongly connected components containing p and q. Both these components are bipartite. we have two different cases. Either q and p belong to a common strongly connected component or not.

Let q and p neither share a component. Again we color the strongly connected components containing p by 1, 2. Also, we color b by 3. Again let x be the cut vertex in a component of p which separates P from q. Without loss of generality assume

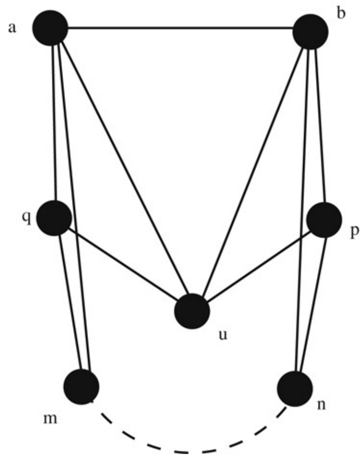

Fig. 4. (a, b, p, u) form an odd subdivision of k_4.

that x is colored by 1. Extend the coloring to entire Z such that the strongly connected components of q are also colored by two colors. If these components are colored by 1, 3, then color a by 2. If these components are colored by 2, 3, then color a by 1. In case these components are colored by 1, 2, then switch the colors 2 and 3 in the component of $G(V - \{x\})$ containing q. Now components of q are colored by 1, 3. Set the color of a to be 2.

Finally consider the case where q and p share a component. In this case we color the strongly connected components of p and q by 1, 2 again. Without loss of generality we assume that q gets color 2. Note that we can recolor all neighboring vertices of q by 3 without harming the rest of the coloring. Extend the coloring to entire Z. This will leave color 1 for a and 3 for b, if $N(q) \cap N(p) = \emptyset$. If not then we cannot color b by 3. In this case (a, b, p, u) form an odd subdivision of K_4 where u is a common neighbor of p and q, see Fig. 4. Note that the segment connecting m, n is of even length and it exists because p and q are at least on one even cycle in Z.

(III) Case of $(b, p) \notin E$ and $(a, q) \notin E$ and $(ab) \in E$ and $col(p) = col(q)$.

In this case, without loss of generality let the colors of both p and q be 1. Let S_1 and S_2 denote the sets of 2-colored and 3-colored vertices respectively, which are adjacent to both, q and p. S_3 and S_4 are sets of 2-colored and 3-colored vertices which are adjacent to p but not q. Similarly S_5 and S_6, which are adjacent to q but not p.

In the following, we will either recolor the original graph (without adding a and b) such that p and q have different colors, or we will find a subdivision of K_4, (a, b, c, d), where these paths may meet only at the end vertices. If we succeed in getting $col(p) \neq col(q)$ then set $col(a) = col(q)$ and $col(b) = col(p)$.

In the following i/j-path denotes an alternating path with colors i and j.

If there is a 2/3 path from $x \in S_1$ to $y \in S_2$ then (a, b, x, y) is a subdivision of K_4. If there is none then in the 2/3 graph we exchange the color of the components which contain vertices of S_2. This reduces S_2 to $\emptyset$. Now either $S_1 = \emptyset$ or not. We consider both these cases separately.

Case 1. $S_1 \neq \emptyset$

If $S_4 = \emptyset$ then the color of p can be changed from 1 to 3. Also, if there is no 2/3 path from any vertex in $S_1 \cup S_3$ to any vertex in S_4, we can change p's color to 3 (after changing colors of components containing S_4 vertices in the 2/3 graph). Furthermore, if there is a 2/3 path from $u \in S_3$ to $v \in S_4$ which does not pass through at least one $w \in S_1$ then (p, b, u, v) is a subdivision of K_4 with $pwab$ being the path from p to a.

Now we have the following, every 2/3 path, if there is any, from any vertex of S_3 to any vertex of S_4 passes through all vertices of S_1. There is at least one vertex in S_4 which has a 2/3 path to some vertex in S_1.

In this case, we do the following, in the 2/3 graph exchange the color of components which contain at least one vertex of S_4 but no vertex of S_1. See that the colors of original S_1 and S_3 vertices will remain unchanged. The new S_4 will be a subset of old S_4 and new S_3 will be a superset of old S_3. Most importantly every vertex of (new) S_4 will have a 2/3 path to S_1, and then new S_4 will be non-empty.

If there is no 1/3 path from S_4 to S_6 then in the 1/3 graph exchange the color of the component containing p. Now p is colored 3 but q is still colored 1.

Otherwise, let there be a 1/3 path, L, from u in S_6 to v in S_4. As shown earlier, every vertex of S_4 has a 2/3 path to S_1. Let M be a 2/3 path from w in S_1 to v in S_4. Clearly $L \cap M$ is non-empty since v belongs to it. Starting from u, let x be the first vertex on L which is also on M. Let M_1 be the section of M from x to w; M_2 be the section of M from x to v; and L_1 be the section of L from x to u.

Observe that x is colored 3, M_1 is an odd length path, and L_1 and M_2 are even length paths. Further, M_1, M_2, and L_1 share only one vertex, namely, x. Then (a, b, w, x) is the subdivision of K_4 where the path from x to w is M_1, the path from x to b is $M_2.b$, and the path from x to a is $L_1.a$.

Case 2. $S_1 = \emptyset$ and $S_2 = \emptyset$
In all these cases we can recolor p or q so that their colors are different.

- There is no 2/3 path from S_3 to S_4: Swap the colors of the 2/3 component containing p, i.e., all color-2 vertices be given color 3 and vice versa in this component.
- There is no 2/3 path from S_5 to S_6: Swap the colors of the 2/3 component containing q.
- There is no 1/2 path from S_3 to S_5: Swap the color of 1/2 component containing p.
- There is no 1/3 path from S_4 to S_6: Swap the color of 1/3 component containing p.

Now if all of the above paths exist, we exchange colors in 2/3 components of the graph (components of the graph induced by vertices colored 2 and 3) containing S_4 vertices but no S_3 vertices. Now every vertex in new S_4 will have a 2/3 path to S_3.

In the new graph still there will be a 1/2 path from some vertex of S_3 to some vertex of S_5 but it is possible that there is no 1/3 path from S_4 to S_6. In that case exchange the colors of the component of 1/3 graph which contains p. Otherwise the graph will have following properties: there will be at least one 2/3 path between S_5 and S_6; at least one 1/2 path between S_3 and S_5; at least one 1/3 path between S_4 and S_6; every S_4 vertex will have a 2/3 path to S_3; and S_3,S_4,S_5,S_6 are all non-empty.

Let M be a path av+1/3 path v to u where v is some vertex of S_6 and u is in S_4. Let N be a 2/3 path from u to some vertex w in S_3. Note this exists because each vertex of S_4 has such a path to S_3. Let L' be a path ay+1/2 path from y to x where y is some vertex in S_5 and x is some vertex in S_3. If $x = w$ then $L = L'$ else $L = L'.p.w$.

Starting from a along L, suppose z is the first vertex common with N (note there has to be such a vertex as w is common); starting from a along M, let r be the first common with N. Starting from z toward a along L, let s be the first vertex which is common with a-to-r section of M (this exists since a is common). Denote s-to-q section of L by L_1; and s-to-r section of M by M_1.

Partition N into 3 parts as follows: if z is nearer w (compared to r) then $bw+$ section of N from w to z is denoted by N_1, $bu+$ the section of N from u to r is denoted by N_3, and the section of N from z to r is denoted by N_2. In case r is closer to w on N, then N_1 is $bu+$ section of N from u to z is N_1, $bw+$ the section of N from w to r is N_3; and the section from z to r is N_2.

Finally let O denote the path constituted by the s-to-a section of $L + ab$, when a is not same as s; otherwise O is ab.

Now we claim (s, b, z, r) is a subdivision of K_4 where the path from b to s is O; from b to z is N_1, b to r is N_3, z to r is N_2, s to z is L_1, and s to r is M_1. See that all paths are odd and they meet only at end-points. □

7. Conclusion

In this paper we generalized the concept of critically indecomposable graphs to X-critical graphs for arbitrary subset X such that $G(X)$ is indecomposable. The main result of this paper shows the existence and uniqueness of a commutative elimination sequence. Using this we showed that in an indecomposable graph a pair of vertices $x, y \in V - X$ can be calculated in $O(n(n+m))$ time such that $G(V - \{x, y\})$ is also indecomposable. We also use commutative elimination sequence to give efficient algorithms for perfect matching and 3-coloring.

Some open problems that need to be addressed next include the discovery of similar structural properties of infinite X-critical graphs along the lines of Schmerl and Trotter [1] and Ille [16]. Also, it will be worthwhile to find an algorithm for testing X-criticality more efficiently than $O(n(n + m))$, the time taken by the brute-force method.

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References

[1] J.H. Schmerl, W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, Discrete Mathematics 113 (1993) 191–205.
[2] P. Bonizzoni, Primitive 2-structures with the $(n - 2)$-property, Theoretical Computer Science 132 (1994) 151–178.
[3] R. Fraïssé, L'intervalle en théorie des relations, ses généralizations, filtre intervallire et clôture d'une relation, in: Orders, Description and Roles, vol. 6, North-Holland, Amsterdam, 1984, pp. 313–342.
[4] R.H. Mohring, H.A. Buer, A fast algorithm for the decomposition of graphs and posets, Mathematics of Operations Research 8 (1983) 170–184.
[5] M. Habib, Substitution des structures combinatoires, théorie et algorithmes, Ph.D. Thesis, Université Pierre et Marie Curie, Paris VI.
[6] D.P. Sumner, Graphs indecomposable with respect to the x-join, Discrete Mathematics 6 (1973) 281–298.
[7] J. Spinrad, p4-trees and substitution decomposition, Discrete Applied Mathematics 39 (1992) 263–291.
[8] R.H. Mohring, F.J. Radermacher, Substitution decomposition for discrete structures and connections with combinatorial optimization, Annals of Discrete Mathematics 19 (1984) 257–356.
[9] A. Ehrenfeucht, G. Rozenberg, Theory of 2-structures, parts i and ii, Theoretical Computer Science 70 (1990) 277–303; 305–342.
[10] A. Ehrenfeucht, G. Rozenberg, Primitivity is hereditary for 2-structures, Theoretical Computer Science 3 (70) (1990) 343–358.
[11] A. Ehrenfeucht, T. Harju, G. Rozenberg, 2-Structures: A Framework for Decomposition and Transformation of Graphs, World Scientific, 1999.
[12] P. Ille, Indecomposable graphs, Discrete Mathematics 173 (1997) 71–78.
[13] A. Cournier, M. Habib, An efficient algorithm to recognize prime undirected graphs, in: E.W. Mayr (Ed.), WG, in: Lecture Notes in Computer Science, vol. 657, Springer, 1992, pp. 212–224.
[14] B. Toft, Problem 10, Academia Praha, New York, 1975.
[15] W. Zang, Proof of toft's conjecture: Every graph containing no fully odd k_4 is 3-colorable, in: W.-L. Hsu, M.-Y. Kao (Eds.), COCOON, in: Lecture Notes in Computer Science, vol. 1449, Springer, 1998, pp. 261–268.
[16] P. Ille, Graphes indécomposables infinis, Comptes Rendus Académie des Sciences. Paris Series I 318 (1994) 499–503.